# Canadian national taper models

by Chhun-Huor Ung[1], Xiao Jing Guo[2] and Mathieu Fortin[3]

**ABSTRACT**

Work was done to gather stem taper data for most forest tree species across Canada. They were used for producing taper models to be applied for the purposes of the national forest inventory and for regional purposes when regional taper models are not available. The models are based on squared DBH and on measured or predicted tree height. A taper equation based on the dimensional analysis approach was adopted to fit Canadian national taper models using the collected data. The model parameters were estimated using a mixed model for taking into account variance heterogeneity and within-tree autocorrelation. In spite of the different protocols for data collection, the accuracy of the proposed stem taper models is similar to that found in previous studies. Consequently, the models seem suitable for pre-harvest estimation of sawlog volume nationally or regionally.

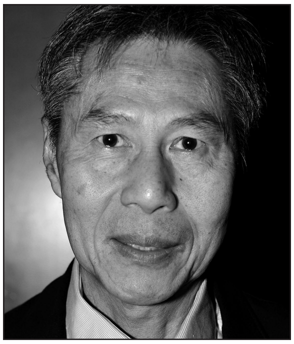
**Keywords:** taper model, mixed model, dimensional analysis approach, national forest inventory

**RÉSUMÉ**

Des travaux ont été réalisés pour collecter les données du défilement de la tige pour la plupart des essences forestières à travers le Canada. Elles ont été utilisées pour produire les modèles de défilement applicables à l'inventaire forestier national ou régional lorsque des modèles de défilement régionaux ne sont pas disponibles. Les modèles sont basés sur le carré du dhp et sur la hauteur de l'arbre mesurée ou prédite. Les paramètres des modèles ont été estimés en utilisant un modèle mixte pour tenir compte de l'hétérogénéité de la variance et de l'autocorrélation intra-arbre. En dépit des différents protocoles de collecte de données, la précision des modèles de défilement de la tige proposés est semblable à celle relevée lors d'études antérieures. Par conséquent, les modèles semblent appropriées pour l'estimation pré-récolte du volume de bois de sciage au niveau national ou régional.

**Mots clés :** modèle de défilement, modèle mixte, approche de l'analyse dimensionnelle, inventaire forestier national.

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## Introduction

Information on wood resources and their economic value is required for decision-making not only in forest management, but also in the wood supply chain. The volume and proportions of the potential timber assortments that can be extracted from the forest stand are involved in determining the economic value of the stand. During and after tree harvesting, timber assortment volumes can be accurately estimated. Currently, harvesters are equipped with a taper measurement device that provides a stream of stem taper data. Another option may be taking standard field measurements of log tapers or using a terrestrial LIDAR scanner. However, before harvesting, i.e., at the beginning of the wood supply chain, this information has to be derived from regional or national inventory data and stem taper models.

To date in Canada, separate projects have been undertaken by provincial agencies to produce these stem taper models. In eastern Canada, these models are available for a few species only. Sharma and Zhang (2004) published stem taper models for jack pine (*Pinus banksiana* Lamb.), black spruce (*Picea mariana* [Mill.] BSP) and balsam fir (*Abies balsamea* [L.]) while Newnham (1988) fitted a stem taper model for red pine (*Pinus resinosa* Sol.). In western Canada, different studies provide these models for a larger array of species: Newnham (1992) for jack pine, lodgepole pine (*Pinus contorta* var. *latifolia* Engelm.), white spruce (*Picea glauca* [Moench] Voss) and trembling aspen (*Populus tremuloides* Michx.), Flewelling and Raynes (1993) for west-

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ern hemlock (*Tsuga heterophylla* (Raf.) Sarg.), and Kozak (1991) for coastal Douglas-fir (*Pseudotsuga menziesii* [Mirb.] Franco), western red cedar (*Thuja plicata* D. Don), western hemlock and black spruce. Basically, stem taper models are not available for all Canadian provinces and when they are, only a few species are covered. In addition to this disparity in terms of species, the minimum upper diameter, which is the limit of merchantable wood volume, varies in existing taper models from one province to another. This variability hinders the comparisons or combinations of inventory results from Canadian provinces.

Considering that many more stem taper data have become available since then, a research project has been undertaken to compile stem taper data across Canada and produce taper models for the whole country and for most tree species. As in the Canadian national biomass equations (Lambert *et al.* 2005, Ung *et al.* 2008), the application context of these taper models is the national forest inventory, meaning that they rely on DBH with or without tree height for a given species. The objective is to present the resulting models after describing the available data and the adopted method.

## Data

The Canadian taper data set comes from two sources. One of these sources was the ENFOR program (Lambert *et al.* 2005) for Ontario, Quebec, and Yukon. The geographical coverage of the ENFOR program data set was broadened by data from Alberta, British Columbia (BC), Manitoba, and Saskatchewan. Also, additional taper data were received from Ontario. Except for BC data in which only the diameter inside bark (DIB) was measured, the data from all other provinces provided the diameters inside and outside bark (DOB) at specific heights. However, DIB was missing in the data from Quebec and Manitoba (Table 1). Note that the diameter at breast height (DBH) refers to DOB at a height of 1.3 m for all the provinces. The taper data were collected using diverse protocols among provinces. This diversity is reflected in the distances between the cross sections.

Some work was required to harmonize the data from different provinces and territories. For the data from BC, the DOB of the height sections was estimated using a model of the DIB–DOB relationship that was fitted using data from other provinces. This model is annexed to this paper (Appendix A). Cottonwood, alder, larch, yellow pine, white pine, hemlock and cedar were removed from the database because they were too scarce and available for BC only. Also in BC data, the species for the genus spruce was sometimes unavailable. In such cases, spruce was considered to be black spruce. Finally, aspen was considered to be trembling aspen and birch was considered to be white birch when only the genus was available. Table 2 shows the distribution of tree species in the final data set. For model evaluation, we randomly split the data into a 70% partition for the fit and a 30% partition for the validation. A summary of these two partitions is shown in Table 3. The considered species are listed in Appendix B.

## Method

### One-parameter taper equation

Sharma and Oderwald (2001) developed a simple tree taper equation by applying a dimensional analysis approach for loblolly pine (*Pinus taeda* L.), with only tree-level covariates involved. The original equation is:

$$[1]\quad d^2 = dbh^2\,\frac{H-h}{H-h_{dbh}}\left(\frac{h}{h_{dbh}}\right)^{2-\gamma}$$

where $\gamma$ is the only parameter to be estimated, $H$ is the total height (m), $h_{dbh}$ is the breast height, i.e., 1.3 m, $dbh$ is diameter outside bark at breast height (cm), $h$ is the cross-section height (m) measured at different locations along the bole, and $d$ is the diameter outside bark (DOB) (cm) for a given cross section. The formulation of eq. 1 ensures that the DOB at height 1.3 m is equal to the DBH.

Stem taper models are generally based on tree DBH and total height. However, whereas tree DBH is measured for all trees in the plot, height is usually measured on a subsample of trees. Lappi (2006) addressed this issue of unobserved tree height in stem taper modeling using a two-point distribution approach. In this study, we adopted a different approach. We decided to substitute tree height in the model for a nonlinear function that is commonly used as a height–diameter relationship. Consequently, to better suit either situation, we adopted two versions of the taper model depending on the availability of tree height measurements. The first version of the model was fitted with observed heights:

**Table 1. Differences between the measurement protocols among the different sources of stem taper data (figures in parentheses indicate respectively minimum and maximum distances between the cross sections)**

| | ENFOR | | | Others | | | | |
|---|---|---|---|---|---|---|---|---|
| | Ontario | Quebec | Yukon | Alberta | BC | Manitoba | Ontario | Saskatchewan |
| DIB | X | – | X | X | X | – | – | X |
| DOB | X | X | X | X | – | X | X | X |
| Distance between the cross sections (m) | | | | | | | | |
| below 1.3 m | (0.02; 1.15) | (0.02; 0.65) | (1.00; 1.00) | (0.10; 1.00) | (0.02; 1.00) | (0.30; 0.34) | (0.02; 1.15) | (0.20; 1.23) |
| above 1.3 m | (0.10; 4.00) | (0.01; 9.58) | (1.30; 2.00) | (0.10; 7.50) | (0.06; 9.78) | (0.20; 2.60) | (0.10; 4.00) | (0.31; 3.00) |

BC: British Columbia
DIB: diameter inside bark
DOB: diameter outside bark

Table 2. Number of trees per province[a] for each species

| Species | Provinces and territories | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | AB | BC | MB | ON | QC | SK | YT | Total |
| Alpine fir | 21 | – | – | – | – | – | – | 21 |
| Balsam fir | 388 | 4857 | – | 36 | 121 | 74 | – | 5476 |
| Balsam poplar | 388 | – | – | 78 | – | – | – | 466 |
| Basswood | – | – | – | 63 | – | – | – | 63 |
| Beech | – | – | – | 59 | 29 | – | – | 88 |
| Black ash | – | – | – | 15 | 10 | – | – | 25 |
| Black cherry | – | – | – | 54 | – | – | – | 54 |
| Black poplar | – | – | – | – | – | 102 | – | 102 |
| Black spruce | 524 | 5301 | 18 | 30 | 639 | 80 | 100 | 6692 |
| Douglas-fir | 90 | 1729 | – | – | – | – | – | 1819 |
| Eastern hemlock | – | – | – | 119 | 61 | – | – | 180 |
| Eastern white cedar | – | – | – | 67 | 70 | – | – | 137 |
| Eastern white pine | – | – | – | 121 | 37 | – | – | 158 |
| Engelmann spruce | 43 | – | – | – | – | – | – | 43 |
| Hickory | – | – | – | 27 | – | – | – | 27 |
| Jack pine | 243 | – | 39 | 64 | 94 | 120 | – | 560 |
| Largetooth aspen | – | – | – | 69 | – | – | – | 69 |
| Lodgepole pine | 2502 | 2053 | – | – | – | 93 | 65 | 4713 |
| Manitoba maple | – | – | – | – | – | 34 | – | 34 |
| Red ash | – | – | – | 23 | – | – | – | 23 |
| Red maple | – | – | – | 61 | 18 | – | – | 79 |
| Red oak | – | – | – | 115 | – | – | – | 115 |
| Red pine | – | – | – | 425 | 36 | – | – | 461 |
| Red spruce | – | – | – | – | 43 | – | – | 43 |
| Silver maple | – | – | – | 29 | – | – | – | 29 |
| Sugar maple | – | – | – | 87 | 5 | – | – | 92 |
| Tamarack | 18 | – | – | 57 | 64 | 24 | – | 163 |
| Trembling aspen | 2814 | 1061 | 21 | 177 | 28 | 111 | 89 | 4301 |
| White ash | – | – | – | 53 | 11 | – | – | 64 |
| White birch | 18 | 33 | – | 121 | 14 | 81 | – | 267 |
| White elm | – | – | – | 58 | – | 29 | – | 87 |
| White oak | – | – | – | 45 | – | – | – | 45 |
| White spruce | 2286 | – | 5 | 43 | 54 | 261 | 200 | 2849 |
| Yellow birch | – | – | – | 85 | 14 | – | – | 99 |
| Total | 9335 | 15034 | 83 | 2181 | 1348 | 1009 | 454 | 29444 |

[a]AB Alberta; BC British Columbia; MB Manitoba; ON Ontario; QC Québec; SK Saskatchewan; YK Yukon

Table 3. Summary statistics of calibration data set and of validation data set (the figures in parentheses provide the minimum and maximum values)

| Species | Calibration | | | Validation | | |
|---|---|---|---|---|---|---|
| | No. of trees | DBH (cm) | height (m) | No. of trees | DBH (cm) | height (m) |
| Alpine fir | 14 | 20.5 ± 3.5 (14.9; 27.5) | 15.1 ± 2.0 (12.2; 18.8) | 7 | 20.7 ± 3.5 (18.0; 27.4) | 14.7 ± 1.5 (12.8; 17.0) |
| Balsam fir | 3850 | 40.6 ± 15.7 (9.2; 133.5) | 28.1 ± 8.2 (9.6; 59.8) | 1647 | 40.5 ± 15.3 (9.5; 114.6) | 28.2 ± 8.0 (9.6; 59.7) |
| Balsam poplar | 325 | 24.5 ± 8.1 (10.0; 51.8) | 19.4 ± 4.0 (8.9; 30.5) | 141 | 24.4 ± 8.8 (10.4; 53.2) | 19.7 ± 3.8 (12.7; 32.0) |
| Basswood | 46 | 30.5 ± 11.8 (12.9; 53.2) | 19.6 ± 4.4 (9.6; 26.1) | 17 | 30.2 ± 13.6 (12.3; 54.8) | 19.2 ± 3.8 (14.1; 25.1) |
| Beech | 64 | 27.9 ± 8.3 (10.5; 46.3) | 20.1 ± 3.7 (9.7; 26.5) | 27 | 28.4 ± 10.1 (12.6; 44.6) | 19.4 ± 3.3 (12.0; 25.4) |
| Black ash | 16 | 22.1 ± 9.3 (10.7; 41.2) | 16.4 ± 2.4 (12.6; 21.0) | 11 | 20.3 ± 6.4 (11.2; 31.4) | 16.7 ± 2.6 (12.8; 20.3) |
| Black cherry | 39 | 26.8 ± 9.9 (9.5; 44.1) | 18.7 ± 3.5 (8.4; 25.9) | 15 | 25.6 ± 8.9 (10.1; 49.6) | 19.0 ± 4.1 (11.3; 23.4) |

**Table 3. Summary statistics of calibration data set and of validation data set (the figures in parentheses provide the minimum and maximum values) (*continued*)**

| | Calibration | | | Validation | | |
|---|---|---|---|---|---|---|
| Species | No. of trees | DBH (cm) | height (m) | No. of trees | DBH (cm) | height (m) |
| Black poplar | 71 | 29.1 ± 9.7 (12.1; 50.6) | 18.6 ± 3.6 (8.9; 28.0) | 31 | 27.4 ± 14.3 (12.0; 72.4) | 18.0 ± 5.8 (8.9; 31.1) |
| Black spruce | 4773 | 38.6 ± 18.5 (9.0; 164.6) | 27.7 ± 9.4 (5.7; 71.2) | 2065 | 38.5 ± 18.6 (9.3; 141.8) | 27.7 ± 9.4 (9.2; 66.5) |
| Douglas-fir | 1270 | 41.3 ± 10.3 (13.6; 74.1) | 28.8 ± 6.7 (11.6; 52.4) | 549 | 42.0 ± 10.9 (13.6; 83.5) | 28.9 ± 7.0 (9.5; 50.6) |
| Eastern hemlock | 134 | 29.4 ± 11.1 (9.5; 51.4) | 16.9 ± 4.2 (7.7; 26.5) | 60 | 28.0 ± 10.4 (10.4; 50.5) | 16.2 ± 4.2 (8.8; 24.1) |
| Eastern white cedar | 100 | 25.2 ± 10.1 (10.2; 66.2) | 13.7 ± 3.0 (8.3; 21.9) | 45 | 25.3 ± 9.6 (10.6; 53.0) | 13.4 ± 2.9 (7.8; 18.5) |
| Eastern white pine | 111 | 33.2 ± 13.9 (10.4; 68.7) | 20.5 ± 5.7 (6.6; 35.9) | 51 | 31.9 ± 13.7 (13.9; 60.7) | 21.8 ± 6.0 (10.0; 38.5) |
| Engelmann spruce | 29 | 23.4 ± 6.6 (13.8; 35.6) | 17.1 ± 3.9 (11.4; 26.3) | 14 | 22.8 ± 6.6 (13.8; 36.6) | 17.6 ± 4.1 (12.4; 27.4) |
| Hickory | 18 | 24.0 ± 8.2 (11.8; 37.4) | 17.6 ± 4.4 (11.6; 24.3) | 9 | 19.6 ± 5.9 (10.7; 27.6) | 16.9 ± 3.8 (12.5; 23.2) |
| Jack pine | 428 | 22.8 ± 6.9 (9.6; 47.1) | 18.4 ± 3.7 (7.6; 28.2) | 183 | 22.5 ± 7.6 (10.5; 53.1) | 18.2 ± 3.6 (9.1; 27.0) |
| Largetooth aspen | 48 | 19.6 ± 7.0 (11.4; 39.2) | 19.7 ± 3.5 (14.1; 28.9) | 21 | 18.8 ± 4.7 (10.5; 27.1) | 19.8 ± 3.6 (13.5; 26.3) |
| Lodgepole pine | 3301 | 27.1 ± 8.6 (10.4; 70.0) | 21.8 ± 5.6 (9.8; 39.7) | 1412 | 27.3 ± 8.6 (10.3; 64.7) | 21.9 ± 5.7 (9.4; 40.8) |
| Manitoba maple | 25 | 22.5 ± 8.2 (11.6; 38.6) | 12.9 ± 4.1 (4.6; 20.9) | 9 | 22.4 ± 5.5 (16.0; 31.4) | 12.3 ± 3.8 (4.6; 18.5) |
| Red ash | 16 | 22.8 ± 7.0 (12.0; 36.8) | 19.7 ± 3.9 (13.5; 26.7) | 7 | 24.0 ± 9.7 (12.0; 40.2) | 18.7 ± 4.3 (13.9; 25.1) |
| Red maple | 59 | 24.2 ± 8.8 (10.3; 45.2) | 18.3 ± 3.8 (11.2; 25.4) | 25 | 21.6 ± 8.8 (10.8; 38.2) | 17.0 ± 4.0 (10.4; 24.9) |
| Red oak | 78 | 22.9 ± 6.3 (10.1; 38.9) | 17.1 ± 2.9 (11.3; 23.0) | 37 | 23.2 ± 6.4 (10.6; 42.1) | 17.2 ± 3.5 (9.9; 23.0) |
| Red pine | 326 | 24.0 ± 8.8 (10.0; 55.1) | 16.3 ± 4.7 (7.3; 34.2) | 140 | 23.7 ± 8.5 (10.2; 53.2) | 15.9 ± 4.6 (7.1; 34.4) |
| Red spruce | 35 | 24.9 ± 8.0 (10.3; 43.5) | 16.4 ± 2.6 (11.3; 21.1) | 17 | 23.8 ± 7.2 (12.2; 36.7) | 17.0 ± 2.2 (13.0; 21.1) |
| Silver maple | 20 | 26.6 ± 11.4 (10.0; 45.3) | 21.6 ± 4.3 (13.6; 26.4) | 9 | 25.7 ± 8.3 (14.6; 41.1) | 22.1 ± 2.5 (16.9; 25.2) |
| Sugar maple | 67 | 30.2 ± 14.4 (10.6; 57.8) | 19.0 ± 4.1 (9.9; 26.4) | 25 | 27.9 ± 10.7 (11.0; 46.8) | 19.1 ± 3.3 (13.3; 24.4) |
| Tamarack | 119 | 20.6 ± 6.8 (10.7; 39.5) | 17.4 ± 4.1 (9.3; 26.8) | 52 | 21.7 ± 7.5 (10.1; 44.5) | 18.1 ± 4.3 (11.9; 30.5) |
| Trembling aspen | 3035 | 26.0 ± 9.4 (10.1; 69.4) | 21.9 ± 4.8 (9.4; 35.9) | 1286 | 26.6 ± 9.6 (10.4; 65.9) | 22.2 ± 4.8 (10.2; 35.5) |
| White ash | 47 | 26.0 ± 8.8 (10.7; 53.7) | 19.4 ± 2.4 (14.4; 26.0) | 18 | 24.3 ± 9.6 (12.5; 42.0) | 17.7 ± 2.9 (11.8; 21.8) |
| White birch | 190 | 20.6 ± 5.9 (10.8; 36.7) | 17.8 ± 3.1 (11.2; 26.6) | 77 | 20.2 ± 5.8 (10.9; 33.4) | 17.7 ± 3.3 (10.5; 25.3) |
| White elm | 59 | 25.7 ± 11.3 (12.3; 61.3) | 15.2 ± 3.7 (6.0; 24.1) | 28 | 26.3 ± 10.9 (13.5; 55.2) | 15.2 ± 3.8 (7.0; 23.2) |
| White oak | 32 | 31.6 ± 16.9 (11.1; 74.3) | 14.1 ± 4.4 (5.0; 21.5) | 13 | 23.6 ± 9.7 (14.3; 41.4) | 11.7 ± 3.5 (7.4; 19.7) |
| White spruce | 2008 | 28.2 ± 10.2 (9.5; 76.9) | 21.7 ± 5.7 (9.0; 38.4) | 852 | 28.4 ± 10.6 (11.3; 83.5) | 21.9 ± 5.6 (7.6; 37.9) |
| Yellow birch | 70 | 37.7 ± 15.9 (10.4; 70.3) | 20.2 ± 3.7 (10.0; 25.2) | 32 | 31.3 ± 11.2 (14.5; 54.2) | 19.9 ± 3.9 (10.9; 25.6) |

$$[2]\quad d_{ijkm}^2 = dbh_{ijk}^2 \frac{H_{ijk} - h_{ijkm}}{H_{ijk} - 1.3}\left(\frac{h_{ijkm}}{1.3}\right)^{2-\gamma_{ijk}} + \varepsilon_{ijkm}$$

where $\varepsilon_{ijkm}$ are Gaussian error terms and the indices $i$, $j$, $k$, and $m$ stand for the province, the plot, the tree and the cross section, respectively.

The second version was modified to account for context where observed tree heights would not be available. To do this, tree height in the model (eq. 2) was replaced by a power function of DBH:

$$[3]\quad d_{ijkm}^2 = dbh_{ijk}^2 \frac{\theta_{0ij} dbh_{ijk}^{\theta_{1ij}} - h_{ijkm}}{\theta_{0ij} dbh_{ijk}^{\theta_{1ij}} - 1.3}\left(\frac{h_{ijkm}}{1.3}\right)^{2-\gamma_{ijk}} + \varepsilon_{ijkm}.$$

where $\theta_{0ij}$ and $\theta_{1ij}$ are two additional parameters to be estimated. Note that the implicit height–diameter relationship is assumed to be constant for the trees in the same plot.

Model (3) is inconsistent only if its right part is negative. This might be the case if $h_{ijkm} > \theta_{0ij}\, dbh_{ijk}^{\theta_{1ij}}$ and $\varepsilon_{ijkm} < 0$. In the fit partition, such a situation is likely to happen for the upper height sections in the tallest trees. On the other hand, these upper sections are located in the non-merchantable part of the tree and are not of great interest for forest managers. To ensure the consistency of the model, we used all the sections up to the first one that had a DOB smaller than 9.0 cm (25 cm for BC). Consequently, both stem taper models are limited to the merchantable part of the tree.

### Model specification

Ordinary least squares estimators assume that the error terms are independently and normally distributed with homogeneous variance. The mixed-effects model theory makes it possible to relax the assumptions of independence and homogeneous variances. Taper data are naturally considered as correlated data because of the repeated measurements on the same tree. The specification of random effects in addition to a covariance structure for the within-tree error terms in nonlinear mixed-effects models is the most common manner to deal with stem taper data (e.g., Garber and Maguire 2003, Lejeune *et al.* 2009). We used this nonlinear mixed-effects approach to fit the two models.

The general parameters $\theta_{0ij}$, $\theta_{1ij}$, $\gamma_{ijk}$ and  may or may not include random effects. After several preliminary trials, the best random effect specification we obtained was when observed heights are available:

$$[4a]\quad d_{ijkm}^2 = dbh_{ijk}^2 \frac{H_{ijk} - h_{ijkm}}{H_{ijk} - 1.3}\left(\frac{h_{ijkm}}{1.3}\right)^{2-(\beta_2+\delta_{i,2}+\delta_{ij,2}+\delta_{ijk})} + \varepsilon_{ijkm}$$

when observed heights are not available:

$$[4b]\quad d_{ijkm}^2 = dbh_{ijk}^2 \frac{\beta_0 dbh_{ijk}^{\beta_1+\delta_{i,1}+\delta_{ij,1}} - h_{ijkm}}{\beta_0 dbh_{ijk}^{\beta_1+\delta_{i,1}+\delta_{ij,1}} - 1.3}\left(\frac{h_{ijkm}}{1.3}\right)^{2-(\beta_2+\delta_{i,2}+\delta_{ij,2}+\delta_{ijk})} + \varepsilon_{ijkm}$$

where $\beta_0$, $\beta_1$ and $\beta_2$ are fixed-effects parameters, and $\boldsymbol{\delta}_i$, $\boldsymbol{\delta}_{ij}$ and $\delta_{ijk}$ are province, plot, and tree random effects, which are assumed to follow multivariate normal distributions with mean 0, i.e.,

$$\boldsymbol{\delta}_i = (\delta_{i,1}, \delta_{i,2})^T \sim N_2(\mathbf{0}, \boldsymbol{\Psi}_{prov}),\ \boldsymbol{\delta}_{ij} = (\delta_{ij,1}, \delta_{ij,2})^T \sim N_2(\mathbf{0}, \boldsymbol{\Psi}_{plot}),$$
$$\text{and } \delta_{ijk} \sim N(0, \sigma_{tree}^2).$$

In addition to these random effects, we also assumed the vector of within-tree residual errors followed a multivariate normal distribution with mean 0, such that $\boldsymbol{\varepsilon}_{ijk} = (\varepsilon_{ijk1}, \varepsilon_{ijk2}, ..., \varepsilon_{ijks_{ijk}})^T \sim N_{s_{ijk}}(\mathbf{0}, \mathbf{R}_{ijk})$ where $s_{ijk}$ is the number of cross sections in tree $k$ of plot $j$ in province $i$. We can further decompose the variance-covariance matrix $\mathbf{R}_{ijk}$ into a variance and correlation structure component to model the heteroscedasticity and the remaining dependence, i.e., $\mathbf{R}_{ijk} = \mathbf{C}_{ijk}^{1/2}\, \mathbf{D}_{ijk}\, \mathbf{C}_{ijk}^{1/2}$ where $\mathbf{C}_{ijk}$ is a diagonal matrix whose elements are the variance modeled through a variance function and $\mathbf{D}_{ijk}$ is a correlation matrix with a predefined structure. The structure of the correlation matrix $\mathbf{D}_{ijk}$ was modeled either as a first-order continuous autoregressive correlation structure or an autoregressive moving average function when convergence could not be reached with the first-order autoregressive structure (Pinheiro and Bates 2000).

The models were fit using the "nlme" package available in R (Pinheiro *et al.* 2009). Akaike's Information Criterion and Schwarz's Bayesian information criterion (BIC) were used to determine the selection of random effects and variance–covariance structure. The models were fit to each species or species group individually.

### Taper model evaluation

Population-averaged predictions are more complicated to obtain when using a nonlinear mixed-effects model. Predictions conditional on the expectation of the random effects, i.e., **0**, are biased estimates of the population-averaged predictions when the random effects enter the model in a nonlinear manner. Fortin *et al.* (2012) have developed a correction that relies on a Taylor series expansion. Using a second-order expansion around the expectation of the random effects, a population-averaged prediction can be obtained through

$$[5]\quad E[d_{ijkm}^2] \approx f(\mathbf{x}_{ijkm}, \boldsymbol{\beta}, \mathbf{0}) + \frac{1}{2} tr\left(\mathbf{Z}'_{ijkm}\left(\boldsymbol{\Psi}_{prov} + \boldsymbol{\Psi}_{tree} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{tree}^2 \end{bmatrix}\right)\right) \equiv \hat{d}_{ijkm}^2$$

where $tr(\cdot)$ is the trace of the matrix argument and $\mathbf{Z}'_{ijkm}$ is the matrix of the second derivatives of the models with respect to the random effects. The details of this approximation and a numerical example are provided in Appendix C.

Using this correction factor, the validity of the model was assessed through the average bias and the root mean square error (RMSE), which were computed as follows:

$$[6a]\quad Bias = \frac{\sum_i \sum_j \sum_k \sum_m (d_{ijkm}^2 - \hat{d}_{ijkm}^2)}{N}$$

$$[6b]\quad RMSE = \sqrt{\frac{\sum_i \sum_j \sum_k \sum_m (d_{ijkm}^2 - \hat{d}_{ijkm}^2)^2}{N}}$$

where $N$ is the total number of observations. Relative bias and relative RMSE were obtained by dividing the bias and RMSE by the average observed squared diameter.

The two models were compared in terms of bias and RMSE for each tenth of relative height and on the basis of the merchantable volume. Merchantable tree volumes were calculated using Smalian's formula. The volume was defined as the volume above stump height till 9 cm of diameter (25 cm for BC). For Saskatchewan, stump height ranged from 0.1 m to 0.5 m. For Quebec, it was 0.15m; for other provinces, it was 0.3 m above ground.

$$[7]\quad V_{ijk} = \sum_{m} \frac{\pi}{80000}\left(d^2_{ijkm} + d^2_{ijk,m+1}\right)\left(h_{ijk,m+1} - h_{ijkm}\right) \text{ for } \forall d_{ijk,m+1} \geq merchLimit$$

where $V_{ijk}$ is the merchantable volume ($m^3$) of tree $k$ in plot $j$ in province $i$ and *merchLimit* is the merchantable limit. Note that the factor 80 000 is required to ensure a proper conversion of units.

## Results and Discussion

The parameter and fit statistics of the mixed taper model (eq. 4a) are given respectively in Tables 4 and 5. The results for the modified taper model (eq. 4b) are presented in Tables 6 and 7. As discussed in Sharma and Oderwald (2001), the parameter $y$ cannot exceed 3.0 for conifers. From Tables 4 and 6 we can see that for both models the estimated fixed part of this parameter, i.e., $\hat{\beta}_2$, varies from 2.1 to 2.3. The model captures the tree shape well. As $y$ increases, the tree butt swells dramatically, and the effect reduces with height. Fig. 1 illustrates the changing taper of a tree with 20 cm DBH and 18 m height when $y$ varies from 2.1 to 2.3.

To illustrate the fit of models 4a and 4b, predicted values were plotted against observed values. Fig. 2 provides an example of these graphs for the black spruce and white birch taper models. Overall, the fit appeared to be good as there were no major departures from the 1:1 reference line.

Given their broad geographic coverage, black spruce and white birch were selected to present the taper model validation and volume estimation. Tables 8 and 9 compare the per-

**Table 4. Parameter estimates of the taper model based on DBH and measured height (eq. 4a)**

| | Fixed effect parameters | Residual and random effect standard deviation | | | | Covariance parameter estimates | |
|---|---|---|---|---|---|---|---|
| Species | $\hat{\beta}_2$ | $\hat{\text{S}}\text{td}(\delta_{i,2})$ | $\hat{\text{S}}\text{td}(\delta_{ij,2})$ | $\hat{\text{S}}\text{td}(\delta_{ijk})$ | $\hat{\text{S}}\text{td}(\varepsilon_{ijkm})$ | Correlation | Moving average |
| Alpine fir[a] | 2.1751 | – | – | 0.0813 | 0.1236 | 0.2387 | 0.3592 |
| Balsam fir | 2.125 | 0.0263 | 0.0579 | 0.0735 | 21.1467 | 0.0922 | – |
| Balsam poplar | 2.1741 | 0.047 | 0.0303 | 0.0536 | 7.0901 | 0.7868 | – |
| Basswood[a] | 2.1492 | – | – | 0.0915 | 10.1096 | 0.5923 | 0.1445 |
| Beech | 2.1408 | 0.0708 | – | 0.084 | 0.1907 | 0.7611 | – |
| Black ash | 2.1421 | – | – | 0.0709 | 10.5654 | 0.7757 | – |
| Black cherry | 2.1739 | – | – | 0.0001 | 0.418 | 0.3651 | – |
| Black poplar | 2.2069 | – | 0.0459 | 0.0997 | 11.3385 | 0.8642 | – |
| Black spruce | 2.1788 | 0.049 | 0.0823 | 0.1053 | 40.9925 | 0.8269 | – |
| Douglas-fir | 2.1436 | – | 0.0398 | 0.0564 | 36.7828 | 0.1011 | – |
| Eastern hemlock | 2.1407 | 0.0463 | 0.0001 | 0.0668 | 0.4504 | 0.7689 | – |
| Eastern white cedar[a] | 2.2224 | 0.0695 | – | 0.1151 | 0.0457 | 0.3503 | 0.1939 |
| Eastern white pine[a] | 2.2216 | – | – | 0.1049 | 0.368 | 0.7832 | 0.2599 |
| Engelmann spruce[a] | 2.1826 | – | 0.0561 | 0.051 | 7.9861 | 0.621 | 0.3418 |
| Hickory[a] | 2.2484 | – | – | 0.0953 | 5.1071 | 0.4745 | -0.051 |
| Jack pine | 2.1413 | 0.0262 | – | 0.0636 | 0.156 | 0.8364 | – |
| Largetooth aspen | 2.0836 | – | – | 0.0441 | 0.1344 | 0.7335 | – |
| Lodgepole pine | 2.1369 | 0.0394 | 0.0422 | 0.0619 | 0.0566 | 0.8516 | – |
| Manitoba maple | 2.1242 | – | – | 0.0647 | 9.0667 | 0.2889 | – |
| Red Ash [a] | 2.1774 | – | – | 0.0002 | 0.7813 | 0.521 | 0.0328 |
| Red maple | 2.1286 | 0.0242 | – | 0.0699 | 0.3153 | 0.597 | – |
| Red oak | 2.2391 | – | 0.088 | 0.1117 | 10.8028 | 0.7097 | – |
| Red pine | 2.1769 | – | 0.029 | 0.0347 | 0.3053 | 0.7848 | – |
| Red spruce | 2.1221 | – | – | 0.0506 | 1.0425 | 0.6412 | – |
| Silver maple[a] | 2.305 | – | – | 0.1171 | 8.662 | 0.6778 | 0.1005 |
| Sugar maple[a] | 2.1661 | – | – | 0.0997 | 14.055 | 0.5412 | 0.1554 |
| Tamarack | 2.2086 | – | – | 0.0805 | 0.2849 | 0.7941 | – |
| Trembling aspen | 2.1482 | 0.048 | 0.0489 | 0.0672 | 5.6962 | 0.8437 | – |
| White ash | 2.2253 | – | – | 0.0966 | 1.9485 | 0.7164 | – |
| White birch | 2.1957 | – | – | 0.0905 | 0.084 | 0.7687 | – |
| White elm | 2.2583 | – | 0.0515 | 0.1339 | 15.9214 | 0.6837 | – |
| White oak | 2.2393 | – | – | 0.1214 | 0.1824 | 0.4258 | – |
| White spruce | 2.1443 | 0.0351 | 0.0317 | 0.0498 | 7.925 | – | – |
| Yellow birch[a] | 2.3467 | – | – | 0.204 | 25.5956 | 0.686 | 0.2364 |

[a]correlation structure ARMA(1,1). See Pinheiro and Bates (2000: 228) for further details.

**Table 5. Parameter estimates of taper model based on DBH and height (eq. 4a) (*continued*)**

| | | Parameter estimate of the variance function | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Species | Type of function[a] | AB | BC | MB | ON | QC | SK | YT | All provinces |
| Alpine fir | Power | – | – | – | – | – | – | – | 1.7606 |
| Balsam fir | Exp | – | – | – | – | – | – | – | 0.0503 |
| Balsam poplar | Exp | 0.0803 | – | – | 0.116 | – | – | – | – |
| Basswood | Exp | – | – | – | – | – | – | – | 0.0694 |
| Beech | Power | – | – | – | 1.9229 | 1.7852 | – | – | – |
| Black ash | Exp | – | – | – | – | – | – | – | 0.0837 |
| Black cherry | Power | – | – | – | – | – | – | – | 1.7827 |
| Black poplar | Exp | – | – | – | – | – | – | – | 0.0774 |
| Black spruce | Exp | 0.0037 | 0.0451 | 0.0162 | -0.056 | 0.0161 | 0.0175 | -0.0037 | – |
| Douglas-fir | Exp | 0.0207 | 0.0394 | – | – | – | – | – | – |
| Eastern hemlock | Power | – | – | – | 1.6356 | 1.582 | – | – | – |
| Eastern white cedar | Power | – | – | – | 2.2507 | 2.0996 | – | – | – |
| Eastern white pine | Power | – | – | – | 1.7069 | 1.7744 | – | – | – |
| Engelmann spruce | Exp | – | – | – | – | – | – | – | 0.0799 |
| Hickory | Exp | – | – | – | – | – | – | – | 0.1069 |
| Jack pine | Power | 1.7686 | – | 1.9121 | 1.8056 | 1.8826 | 1.9167 | – | – |
| Largetooth aspen | Power | – | – | – | – | – | – | – | 1.7374 |
| Lodgepole pine | Power | – | – | – | – | – | – | – | 2.1996 |
| Manitoba maple | Exp | – | – | – | – | – | – | – | 0.0801 |
| Red ash | Power | – | – | – | – | – | – | – | 1.4159 |
| Red maple | Power | – | – | – | 1.6768 | 1.7039 | – | – | – |
| Red oak | Power | – | – | – | – | – | – | – | 0.7237 |
| Red pine | Power | – | – | – | – | – | – | – | 1.6612 |
| Red spruce | Power | – | – | – | – | – | – | – | 1.2235 |
| Silver maple | Exp | – | – | – | – | – | – | – | 0.0824 |
| Sugar maple | Exp | – | – | – | 0.0642 | 0.0815 | – | – | – |
| Tamarack | Power | 1.6504 | – | – | 1.7514 | 1.6576 | 1.6585 | – | – |
| Trembling aspen | Exp | – | – | – | – | – | – | – | 0.0952 |
| White ash | Power | – | – | – | 1.2585 | 1.2237 | – | – | – |
| White birch | Power | 1.9352 | 2.0578 | – | 2.0704 | 2.2451 | 2.2039 | – | – |
| White elm | Exp | – | – | – | 0.0639 | – | 0.0646 | – | – |
| White oak | Power | – | – | – | – | – | – | – | 1.9209 |
| White spruce | Exp | 0.0679 | – | 0.0583 | 0.0695 | 0.0732 | 0.0798 | 0.0597 | – |
| Yellow birch | Exp | – | – | – | – | – | – | – | 0.0582 |

[a]Power: Power function, Exp: Exponential function. See Pinheiro and Bates (2000: 218) for further details.

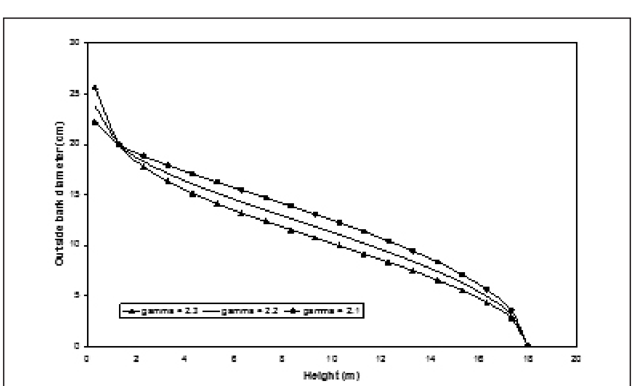


**Fig. 1.** Changing taper with parameter $\beta_2$ values of eq. 4a

formance of two taper models by relative height classes. For black spruce, the model with the observed height (eq. 4a) performs better than the model without the observed height (eq. 4b). Both models are equivalent for white birch. The relationship between DBH and height (H–D relationship) may explain this result. It is well recognized that for shade-tolerant species such as black spruce, this relationship varies over time (Messier *et al*. 1999, Varga *et al*. 2005). In fact, after remaining as undergrowth for a long time, black spruce trees may start to grow as young saplings. Also, black spruce can grow on a very large site gradient from poor to rich sites. This leads to two effects: the development rate of black spruce can vary largely between stands and its age of maturity can vary greatly from stand to stand. For this reason, several studies linked the development of an H–D relationship with local and regional

**Table 6. Parameter estimates and fitting statistics of taper model based on DBH and predicted height (eq. 4b)**

| | Fixed effect parameters | | | Residual and random effect standard deviation | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Species | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\text{S}}\text{td}(\delta_{i,1})$ | $\hat{\text{S}}\text{td}(\delta_{ij,1})$ | $\hat{\text{S}}\text{td}(\delta_{i,2})$ | $\hat{\text{S}}\text{td}(\delta_{ij,2})$ | $\hat{\text{S}}\text{td}(\delta_{ijk})$ | $\hat{\text{S}}\text{td}(\varepsilon_{ijkm})$ |
| Alpine fir | 4.8689 | 0.3822 | 2.1805 | – | – | – | – | 0.0823 | 4.0238 |
| Balsam fir | 10.4602 | 0.2401 | 2.211 | 0.0913 | – | – | 0.0780 | 0.0872 | 34.7979 |
| Balsam poplar | 6.0469 | 0.3672 | 2.1855 | – | 0.0499 | 0.0569 | – | 0.0642 | 7.0681 |
| Basswood | 10.1134 | 0.1957 | 2.1507 | – | 0.0155 | – | – | 0.1035 | 11.048 |
| Beech | 10.8759 | 0.239 | 2.1514 | – | 0.0285 | 0.0731 | – | 0.0790 | 16.4041 |
| Black ash | 9.3651 | 0.3037 | 2.1589 | – | 0 | – | – | 0.0701 | 0.0965 |
| Black cherry | 204.8406 | -0.7245 | 2.1872 | – | 0.0567 | – | – | 0.0001 | 0.3162 |
| Black poplar | 2.9195 | 0.5727 | 2.2118 | – | – | – | – | 0.1092 | 10.7518 |
| Black spruce | 15.7745 | 0.15 | 2.2548 | 0.1639 | – | – | 0.0746 | 0.0816 | 23.1558 |
| Douglas-fir | 504.288 | -0.7858 | 2.1973 | 0.2014 | 0.0778 | – | – | 0.0661 | 22.1244 |
| Eastern hemlock | 5.9068 | 0.3675 | 2.1633 | 0.0284 | 0.0496 | – | – | 0.0774 | 0.3495 |
| Eastern white cedar | 10.6555 | 0.1222 | 2.2283 | – | 0.0457 | 0.0700 | – | 0.0993 | 0.0594 |
| Eastern white pine | 13.0372 | 0.1932 | 2.2333 | – | 0.0711 | – | – | 0.1045 | 0.3995 |
| Engelmann spruce | 1.9517 | 0.7223 | 2.1812 | – | – | – | 0.0584 | 0.0490 | 7.9718 |
| Hickory | 6.227 | 0.3651 | 2.2766 | – | – | – | – | 0.1168 | 8.9391 |
| Jack pine | 15.5003 | 0.0654 | 2.1485 | – | 0.0675 | 0.0250 | – | 0.0636 | 6.8989 |
| Largetooth aspen | 10.3104 | 0.2124 | 2.0805 | – | 0.0394 | – | – | 0.0448 | 0.1447 |
| Lodgepole pine | 11.8367 | 0.246 | 2.1264 | – | 0.0953 | 0.0330 | – | 0.0000 | 0.0763 |
| Manitoba maple | 5.6077 | 0.2909 | 2.137 | – | 0.0001 | – | – | 0.0553 | 8.1632 |
| Red ash | 26.6285 | -0.0106 | 2.1971 | – | – | – | – | 0.0001 | 0.6272 |
| Red maple | 7.8918 | 0.254 | 2.1212 | 0.0927 | 0.0977 | – | – | 0.0743 | 0.241 |
| Red oak | 44.7444 | -0.2235 | 2.2605 | – | – | – | – | 0.1269 | 9.5911 |
| Red pine | 20.296 | -0.0353 | 2.1781 | – | 0.1358 | – | – | 0.0449 | 0.1837 |
| Red spruce | 20.6225 | 0.0091 | 2.1352 | – | – | – | – | 0.0476 | 0.7719 |
| Silver maple | 201.2468 | -0.5842 | 2.3036 | – | – | – | – | 0.0000 | 8.9144 |
| Sugar maple | 9.7271 | 0.2095 | 2.1706 | – | 0.0278 | – | – | 0.1115 | 17.3426 |
| Tamarack | 59.2599 | -0.3284 | 2.2219 | 0.0617 | 0.0432 | – | – | 0.0771 | 6.6109 |
| Trembling aspen | 6.7233 | 0.4554 | 2.1583 | 0.2207 | – | – | 0.0699 | 0.0653 | 6.3263 |
| White ash | 71.2866 | -0.3535 | 2.2328 | – | 0.0004 | – | – | 0.0968 | 1.5239 |
| White birch | 22.143 | 0.0233 | 2.2069 | 0.1158 | – | – | – | 0.0874 | 0.0695 |
| White elm | 2.3524 | 0.5602 | 2.2465 | – | 0 | – | – | 0.1449 | 15.0438 |
| White oak | 10.3196 | 0.0802 | 2.2391 | – | – | – | – | 0.1030 | 31.38 |
| White spruce | 5.7006 | 0.4099 | 2.1581 | – | 0.0649 | 0.0452 | – | 0.0631 | 0.0465 |
| Yellow birch | 30.7651 | -0.003 | 2.3058 | – | – | – | – | 0.1316 | 23.1983 |

variables. Bégin and Raulier (1995) succeeded in increasing the accuracy of prediction of H by adding to D the mean diameter and mean height of the sampled trees in the plots. Fortin *et al.* (2009) were able to reduce the bias in the prediction of H by adding to D not only the stand basal area but also the annual mean temperature, plot drainage, social status and the occurrence of disturbance. As white birch is a shade-intolerant species, its H–D relationship varies less than for black spruce. This slight variation may explain why eq. 4a and eq. 4b are equivalent for white birch. Indeed, the estimation of $\beta_2$ for both models is very close: 2.1957 for eq. 4a, and 2.2069 for eq. 4b. The results of volume comparison are consistent with the results of diameter validation (Table 10). For both species and both models, positive and negative biases occur in any height class, but the positive bias prevails. The predominance of positive bias indicates that the model tends to underpredict. This underestimation is understandable in the national context of this work, for which local data on forest stands are not available.

## Conclusion

The Canadian national taper models were developed from taper data provided by provincial and territorial agencies. These models are based on the dimensional analysis approach. Their parameters were estimated using the nonlinear mixed-effects model. The tree DBH with or without height represents their input. They have been developed for national applications as well as for provincial and territorial

**Table 7. Parameter estimates and fitting statistics of taper model based on DBH and predicted height (eq. 4b) (continued)**

| | | Parameter of the variance function ($\theta_i$) | | | | | | | | Covariance parameter estimates | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Species** | **Type of function**[b] | **AB** | **BC** | **MB** | **ON** | **QC** | **SK** | **YT** | | **Corre-lation** | **Moving average** |
| Alpine fir[a] | Exp | – | – | – | – | – | – | – | 0.0921 | 0.1894 | 0.4844 |
| Balsam fir | Exp | 0.0202 | 0.0447 | – | -0.0257 | 0.0303 | 0.0258 | – | – | 0.8255 | – |
| Balsam poplar | Exp | 0.0804 | – | – | 0.1158 | – | – | – | – | 0.7848 | – |
| Basswood[a] | Exp | – | – | – | – | – | – | – | 0.0686 | 0.6514 | 0.129 |
| Beech | Exp | – | – | – | 0.0681 | 0.0469 | – | – | – | 0.7520 | – |
| Black ash | Power | – | – | – | 2.1632 | 2.0541 | – | – | – | 0.7008 | – |
| Black cherry | Power | – | – | – | – | – | – | – | 1.8518 | 0.2601 | – |
| Black poplar | Exp | – | – | – | – | – | – | – | 0.0767 | 0.8439 | – |
| Black spruce | Exp | 0.0299 | 0.0492 | 0.0122 | -0.0347 | 0.0168 | 0.034 | 0.0234 | – | 0.0707 | – |
| Douglas-fir | Exp | 0.0402 | 0.0472 | – | – | – | – | – | – | 0.1325 | – |
| Eastern hemlock | Power | – | – | – | – | – | – | – | 1.6971 | 0.7557 | – |
| Eastern white cedar | Power | – | – | – | – | – | – | – | 2.0957 | – | – |
| Eastern white pine[a] | Power | – | – | – | 1.6828 | 1.7229 | – | – | – | 0.7683 | 0.2747 |
| Engelmann spruce | Exp | – | – | – | – | – | – | – | 0.081 | 0.8534 | – |
| Hickory | Exp | – | – | – | – | – | – | – | 0.0914 | 0.6596 | – |
| Jack pine | Exp | 0.0742 | – | 0.0905 | 0.0758 | 0.0823 | 0.0881 | – | – | 0.8159 | – |
| Largetooth aspen | Power | – | – | – | – | – | – | – | 1.7285 | 0.7626 | – |
| Lodgepole pine | Power | – | – | – | – | – | – | – | 2.0461 | 0.1791 | – |
| Manitoba maple | Exp | – | – | – | – | – | – | – | 0.0891 | 0.4276 | – |
| Red ash[a] | Power | – | – | – | – | – | – | – | 1.4677 | 0.3991 | 0.0714 |
| Red maple[a] | Power | – | – | – | – | – | – | – | 1.7089 | 0.6454 | -0.1158 |
| Red oak | Power | – | – | – | – | – | – | – | 0.7628 | 0.6958 | – |
| Red pine | Power | – | – | – | – | – | – | – | 1.7604 | 0.6709 | – |
| Red spruce | Power | – | – | – | – | – | – | – | 1.2523 | 0.3471 | – |
| Silver maple | Exp | – | – | – | – | – | – | – | 0.0831 | 0.5178 | 0.1849 |
| Sugar maple | Exp | – | – | – | – | – | – | – | 0.0612 | 0.7496 | – |
| Tamarack | Exp | 0.0869 | – | – | 0.0946 | 0.0795 | 0.0838 | – | – | 0.7417 | – |
| Trembling aspen | Exp | – | – | – | – | – | – | – | 0.0901 | 0.8304 | – |
| White ash | Power | – | – | – | – | – | – | – | 1.3246 | 0.7005 | – |
| White birch | Power | 2.0355 | 2.0901 | – | 2.1253 | 2.2697 | 2.2564 | – | – | 0.7484 | – |
| White elm | Exp | – | – | – | – | – | – | – | 0.0666 | 0.7025 | – |
| White oak | Exp | – | – | – | – | – | – | – | 0.0453 | 0.5022 | – |
| White spruce | Power | 2.138 | – | 2.0165 | 2.1283 | 2.1319 | 2.2604 | 2.0672 | – | – | – |
| Yellow birch | Exp | – | – | – | 0.0587 | 0.0455 | – | – | – | – | – |

[a]correlation structure ARMA(1,1), see Pinheiro and Bates (2000: 228) for further details
[b]Power: Power function, Exp: Exponential function, see Pinheiro and Bates (2000: 218) for further details.

purposes, when necessary. Their main limitation is the relatively low number of diameter measurements along the tree boles available for their calibration. However, the accuracy of the wood volume derived from the proposed models is similar to that of existing equations. This result indicates that the proposed taper models are suitable for national applications as well as for regional ones when regional models are not available.

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**Table 8. Bias and RMSE of the validation data set of DOB for black spruce**

| Model | Relative height | N | Bias (cm$^2$) | RMSE (cm$^2$) |
|---|---|---|---|---|
| Eq. 4a | 0.0 < h/H < 0.1 | 6683 | 222 (9%) | 916 (35%) |
| | 0.1 < h/H < 0.2 | 1872 | 3 (0%) | 349 (25%) |
| | 0.2 < h/H < 0.3 | 1684 | 57 (5%) | 334 (27%) |
| | 0.3 < h/H < 0.4 | 1521 | 81 (7%) | 268 (24%) |
| | 0.4 < h/H < 0.5 | 1365 | 67 (7%) | 247 (25%) |
| | 0.5 < h/H < 0.6 | 1054 | 32 (4%) | 244 (27%) |
| | 0.6 < h/H < 0.7 | 770 | -33 (-5%) | 245 (33%) |
| | 0.7 < h/H < 0.8 | 376 | -134 (-22%) | 317 (52%) |
| | 0.8 < h/H < 0.9 | 90 | -293 (-57%) | 433 (85%) |
| | 0.9 < h/H < 1.0 | 3 | -212 (-84%) | 296 (118%) |
| | Overall | 15418 | 112 (6%) | 643 (37%) |
| Eq. 4b | 0.0 < h/H < 0.1 | 6683 | 69 (3%) | 796 (31%) |
| | 0.1 < h/H < 0.2 | 1872 | 230 (16%) | 437 (31%) |
| | 0.2 < h/H < 0.3 | 1684 | 379 (31%) | 643 (52%) |
| | 0.3 < h/H < 0.4 | 1521 | 487 (43%) | 843 (75%) |
| | 0.4 < h/H < 0.5 | 1365 | 554 (56%) | 967 (98%) |
| | 0.5 < h/H < 0.6 | 1049 | 656 (74%) | 1092 (124%) |
| | 0.6 < h/H < 0.7 | 715 | 580 (91%) | 825 (129%) |
| | 0.7 < h/H < 0.8 | 246 | 542 (125%) | 766 (177%) |
| | 0.8 < h/H < 0.9 | 24 | 52 (51%) | 242 (237%) |
| | 0.9 < h/H < 1.0 | 1 | -67 (-54%) | 67 (54%) |
| | Overall | 15160 | 281 (16%) | 793 (45%) |

**Table 9. Bias and RMSE of the validation data set of DOB for white birch**

| Model | Relative height | N | Bias(cm$^2$) | RMSE(cm$^2$) |
|---|---|---|---|---|
| Eq. 4a | 0.0 < h/H < 0.1 | 184 | -4 (-1%) | 91 (15%) |
| | 0.1 < h/H < 0.2 | 86 | 40 (11%) | 84 (22%) |
| | 0.2 < h/H < 0.3 | 79 | 48 (15%) | 72 (23%) |
| | 0.3 < h/H < 0.4 | 68 | 51 (19%) | 70 (26%) |
| | 0.4 < h/H < 0.5 | 73 | 40 (19%) | 68 (32%) |
| | 0.5 < h/H < 0.6 | 53 | 19 (12%) | 42 (26%) |
| | 0.6 < h/H < 0.7 | 39 | 7 (6%) | 32 (24%) |
| | 0.7 < h/H < 0.8 | 23 | -14 (-16%) | 25 (28%) |
| | 0.8 < h/H < 0.9 | 1 | -42 (-103%) | 42 (103%) |
| | Overall | 606 | 23 (6%) | 74 (21%) |
| Eq. 4b | 0.0 < h/H < 0.1 | 184 | -6 (-1%) | 92 (16%) |
| | 0.1 < h/H < 0.2 | 86 | 38 (10%) | 83 (22%) |
| | 0.2 < h/H < 0.3 | 79 | 41 (13%) | 72 (22%) |
| | 0.3 < h/H < 0.4 | 68 | 38 (14%) | 66 (25%) |
| | 0.4 < h/H < 0.5 | 73 | 21 (10%) | 61 (29%) |
| | 0.5 < h/H < 0.6 | 53 | -5 (-3%) | 45 (28%) |
| | 0.6 < h/H < 0.7 | 39 | -19 (-15%) | 42 (32%) |
| | 0.7 < h/H < 0.8 | 23 | -48 (-55%) | 60 (68%) |
| | 0.8 < h/H < 0.9 | 1 | -39 (-94%) | 39 (94%) |
| | Overall | 606 | 12 (3%) | 75 (21%) |

**Table 10. Comparison of volume**

| Species | N | Model | Bias (m$^3$) | RMSE (m$^3$) |
|---|---|---|---|---|
| Black spruce | 2065 | eq. 4a | 0.058 (3.4%) | 0.535 (30.9%) |
| | | eq. 4b | 0.624 (37.6%) | 1.234 (74.3%) |
| White birch | 77 | eq. 4a | 0.035 (15.0%) | 0.052 (22.1%) |
| | | eq. 4b | 0.027 (11.3%) | 0.051 (21.6%) |

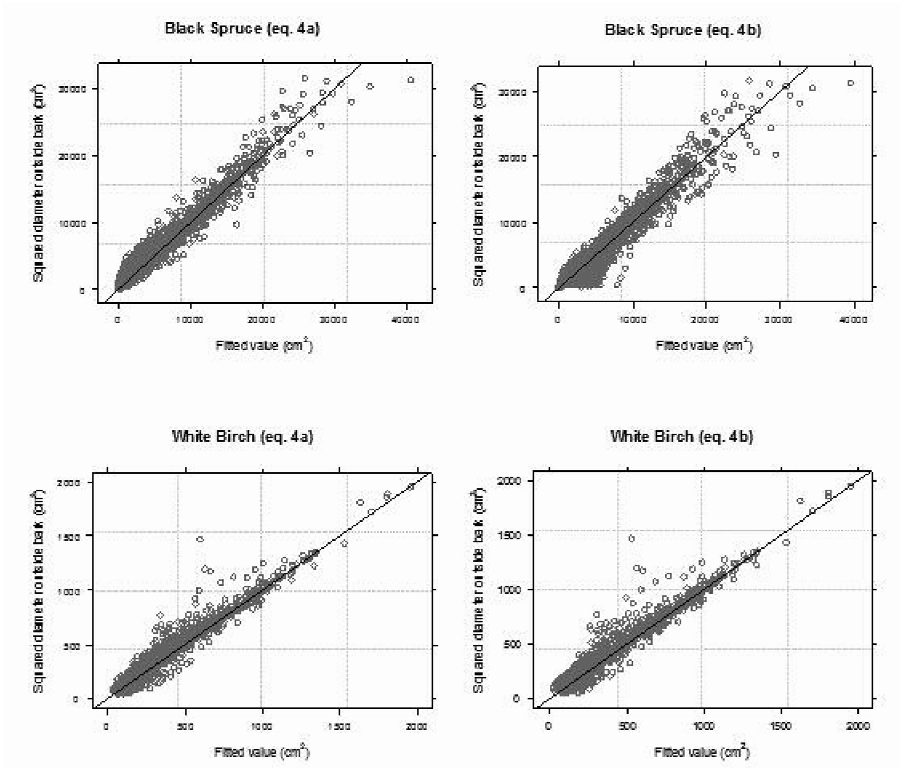


**Fig. 2.** Model fitting for both eq. 4a and eq. 4b using black spruce as an example.

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# Appendices

## Appendix A. Model fitting for the diameter inside-outside bark model

Since diameter outside bark is missing from the BC data, a simple linear model was fitted with the data from provinces where both the diameter inside bark and diameter outside bark were available. However, these diameters are correlated data because they resulted from repeated measurements on the same trees. As usual, we considered a series of mixed models with different types of variance–covariance structures and random effects. Moreover, for most species, heteroscedasticity was detected. Residual variance was modeled as a power-of-the-mean function with its parameters being different across the provinces. The model with the smallest Akaike Information Criteria (AIC) is presented below:

$$d_{ijkm} = (\beta_0 + u_{ijk,1}) + (\beta_1 + u_{ijk,2})\, dib_{ijkm} + \varepsilon_{ijkm} \quad \text{(A1)}$$

where $d_{ijkm}$ is the diameter outside bark (cm) of cross section $m$ of tree $k$ in plot $j$ in province $i$, $dib_{ijkm}$ is the diameter inside bark (cm); $\beta_0$ and $\beta_1$ are fixed effects; $u_{ijk,0}$ and $u_{ijk,1}$ are two tree-level random effects with mean **0** and variance-covariance **G**, such that $\mathbf{u}_{ijk} = (u_{ijk,0},\ u_{ijk,1})^T \sim N_2(\mathbf{0},\mathbf{G})$; and $\varepsilon_{ijkm}$ is the residual error term with mean 0 and $Var(\varepsilon_{ijkm}) = \sigma^2\, \hat{d}_{ijkm}^{\,\delta_i}$, $\hat{d}_{ijkm}$ is the prediction of $d_{ijkm}$ from fixed effects only and $\delta_i$ is a province-specific variance parameter to be estimated. PROC MIXED (SAS Institute Inc. 2008) was used to calibrate the model.

**Table A1. Model fitting for the diameter inside-outside bark model: fixed effects (standard errors appear in parentheses)**

| Species name | $\hat{\beta}_0$ | $\hat{\beta}_1$ |
|---|---|---|
| Balsam fir | 0.3300 (0.0073) | 1.0443 (0.0008) |
| Black spruce | 0.3832 (0.0049) | 1.0347 (0.0006) |
| Douglas-fir | 0.2374 (0.0165) | 1.0953 (0.0025) |
| Lodgepole pine | 0.2659 (0.0035) | 1.0271 (0.0004) |
| Trembling aspen | 0.2183 (0.0017) | 1.0639 (0.0004) |
| White birch | 0.1380 (0.0070) | 1.0619 (0.0010) |

**Table A2. Model fitting for the diameter inside-outside bark model: random effects (standard errors appear in parentheses)**

| Species | Province | $\hat{G}_{11}$ | $\hat{G}_{12}$ | $\hat{G}_{22}$ | $\hat{\sigma}^2$ | $\hat{\delta}_i$ |
|---|---|---|---|---|---|---|
| Balsam fir | – | 0.0217 (0.0020) | -0.0007 (0.0002) | 0.0003 (0.0000) | 0.0027 (0.0002) | – |
| | AB | – | – | – | – | 1.1407 (0.0326) |
| | ON | – | – | – | – | 0.4744 (0.0475) |
| | SK | – | – | – | – | 1.1182 (0.0390) |
| Black spruce | – | 0.0227 (0.0014) | -0.0012 (0.0001) | 0.0003 (0.0000) | 0.0050 (0.0003) | – |
| | AB | – | – | – | – | 0.7094 (0.0272) |
| | ON | – | – | – | – | 0.3330 (0.0426) |
| | SK | – | – | – | – | 0.9170 (0.0307) |
| | YT | – | – | – | – | 0.9507 (0.0313) |
| Douglas-fir | – | 0.0130 (0.0037) | -0.0011 (0.0005) | 0.0003 (0.0001) | 0.0005 (0.0001) | – |
| | AB | – | – | – | – | 2.2442 (0.0869) |
| Lodgepole pine | – | 0.0142 (0.0010) | -0.0008 (0.0001) | 0.0003 (0.0000) | 0.0115 (0.0005) | – |
| | AB | – | – | – | – | 0.5865 (0.0157) |
| | SK | – | – | – | – | 0.6676 (0.0179) |
| | YT | – | – | – | – | 0.3943 (0.0275) |
| | ON | – | – | – | – | 0.6995 (0.0406) |
| Trembling aspen | – | 0.0004 (0.0000) | – | – | 0.0068 (0.0001) | – |
| | AB | – | – | – | – | 1.0644 (0.0074) |
| | ON | – | – | – | – | 0.8924 (0.0139) |
| | SK | – | – | – | – | 1.0956 (0.0128) |
| | YT | – | – | – | – | 1.0546 (0.0200) |
| White birch | – | 0.0113 (0.0014) | -0.0001 (0.0001) | 0.0003 (0.0000) | 0.0102 (0.0006) | – |
| | AB | – | – | – | – | 0.2940 (0.0428) |
| | ON | – | – | – | – | 0.4545 (0.0256) |
| | SK | – | – | – | – | 0.7310 (0.0244) |

**Appendix B. Species list**

| | | |
|---|---|---|
| Alpine fir | Sapin subalpin | *Abies lasiocarpa* (Hook.) Nutt. |
| Balsam fir | Sapin baumier | *Abies balsamea* (L.) Mill. |
| Balsam poplar | Peuplier baumier | *Populus balsamifera* L. |
| Basswood | Tilleul d'Amérique | *Tilia americana* L. |
| Beech | Hêtre à grandes feuilles | *Fagus grandifolia* Ehrh. |
| Black ash | Frêne noir | *Fraxinus nigra* Marsh. |
| Black cherry | Cerisier tardif | *Prunus serotina* Ehrh. |
| Black spruce | Épinette noire | *Picea mariana* (Mill.) BSP |
| Black poplar | Peuplier noir | *Populus nigra* L. |
| Douglas-fir | Douglas vert | *Pseudotsuga menziesii* var. *menziesii* (Mirb.) Franco |
| Eastern hemlock | Pruche du Canada | *Tsuga canadensis* (L.) Carr. |
| Eastern redcedar | Genévrier rouge | *Juniperus virginiana* L. |
| Eastern white pine | Pin blanc | *Pinus strobus* L. |
| Engelmann spruce | Épinette d'Engelmann | *Picea engelmannii* Parry ex Engelm. |
| Hickory | Caryer | *Carya* Nutt. |
| Jack pine | Pin gris | *Pinus banksiana* Lamb. |
| Largetooth aspen | Peuplier à grandes dents | *Populus grandidentata* Michx. |
| Lodgepole pine | Pin tordu latifolié | *Pinus contorta* var. *latifolia* Engelm. |
| Manitoba maple | Érable à feuilles composées | *Acer negundo* L. |
| Red maple | Érable rouge | *Acer rubrum* L. |
| Red oak | Chêne rouge | *Quercus rubra* L. |
| Red pine | Pin rouge | *Pinus resinosa* Ait. |
| Red spruce | Épinette rouge | *Picea rubens* Sarg. |
| Silver maple | Érable argenté | *Acer saccharinum* L. |
| Sugar maple | Érable à sucre | *Acer saccharum* Marsh. |
| Tamarack larch | Mélèze laricin | *Larix laricina* (Du Roi) K. Koch |
| Trembling aspen | Peuplier faux-tremble | *Populus tremuloides* Michx. |
| White ash | Frêne blanc | *Fraxinus americana* L. |
| White birch | Bouleau à papier | *Betula papyrifera* Marsh. |
| White elm | Orme d'Amérique | *Ulmus americana* L. |
| White oak | Chêne blanc | *Quercus alba* L. |
| White spruce | Épinette blanche | *Picea glauca* (Moench) Voss |
| Yellow birch | Bouleau jaune | *Betula alleghaniensis* Britton (*Betula lutea* Michx. F.) |

## Appendix C. Computing population-averaged predictions from stem taper models.

Let us consider a stem taper model with only one level of random effects such that $d^2_{ijkm} = f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i) + \varepsilon_{ijkm}$ with $\mathbf{x}_{ijkm}$ and $\boldsymbol{\beta}$ being the vector of covariates and the vector of fixed-effects parameters respectively. This model can be approximated through a second-order Taylor expansion as follows:

$$\text{(C1)}\qquad d^2_{ijkm} = f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i) + \varepsilon_{ijkm} \approx f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\mathbf{0}) + \mathbf{z}_{ijkm}\boldsymbol{\delta}_i + \frac{1}{2}\boldsymbol{\delta}_i^T\mathbf{Z}'_{ijkm}\boldsymbol{\delta}_i + \varepsilon_{ijkm}$$

where $\mathbf{z}_{ijkm} = \left.\dfrac{\partial f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i)}{\partial \boldsymbol{\delta}_i^T}\right|_{\boldsymbol{\delta}_i=\mathbf{0}}$ and $\mathbf{Z}'_{ijkm} = \left.\dfrac{\partial^2 f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i)}{\partial \boldsymbol{\delta}_i \partial \boldsymbol{\delta}_i^T}\right|_{\boldsymbol{\delta}_i=\mathbf{0}}$.

From the approximation C1, the population-averaged prediction of the squared diameter can be derived as

$$\text{(C2)}\qquad E[d^2_{ijkm}] \approx f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\mathbf{0}) + \frac{1}{2}tr(\mathbf{Z}'_{ijkm}\boldsymbol{\Psi})$$

where $tr(\cdot)$ is the trace of the matrix argument and $\boldsymbol{\Psi}$ is the variance-covariance matrix of the random-effect parameters $\boldsymbol{\delta}_i$.

The elements of matrix $\mathbf{Z}'_{ijkm}$ are given by

$$\text{(C3a)}\qquad \left.\frac{\partial^2 f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i)}{\partial^2 \delta_{i,1}}\right|_{\boldsymbol{\delta}_i=\mathbf{0}} = dbh^2_{ijk}\left(\frac{h_{ijkm}}{1.3}\right)^{2-\beta_2}\frac{\ln^2(dbh_{ijk})\beta_0 dbh_{ijk}^{\beta_1}(h_{ijkm}-1.3)(-\beta_0 dbh_{ijk}^{\beta_1}-1.3)}{(\beta_0 dbh_{ijk}^{\beta_1}-1.3)^3}$$

$$\text{(C3b)}\qquad \left.\frac{\partial^2 f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i)}{\partial \delta_{i,1}\partial \delta_{i,2}}\right|_{\boldsymbol{\delta}_i=\mathbf{0}} = -dbh^2_{ijk}\left(\frac{h_{ijkm}}{1.3}\right)^{2-\beta_2}\frac{\ln(dbh_{ijk})\beta_0 dbh_{ijk}^{\beta_1}(h_{ijkm}-1.3)}{(\beta_0 dbh_{ijk}^{\beta_1}-1.3)^2}\ln\left(\frac{h_{ijkm}}{1.3}\right)$$

$$\text{(C3c)}\qquad \left.\frac{\partial^2 f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\boldsymbol{\delta}_i)}{\partial^2 \delta_{i,2}}\right|_{\boldsymbol{\delta}_i=\mathbf{0}} = f(\mathbf{x}_{ijkm},\boldsymbol{\beta},0)\ln^2\left(\frac{h_{ijkm}}{1.3}\right)$$

Extending this approach to the three levels of random effects in this study, we obtain:

$$\text{(C4)}\qquad \hat{d}^2_{ijkm} = f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\mathbf{0}) + \frac{1}{2}tr\left(\mathbf{Z}'_{ijkm}\left(\boldsymbol{\Psi}_{prov} + \boldsymbol{\Psi}_{plot} + \begin{bmatrix}0 & 0\\ 0 & \sigma^2_{tree}\end{bmatrix}\right)\right)$$

In practice, the variance-covariance of the random effects is unknown and $\boldsymbol{\Psi}_{prov}$, $\boldsymbol{\Psi}_{plot}$, and $\sigma^2_{tree}$ are replaced by the maximum likelihood estimates.

Considering the black spruce model with $dbh_{ijk}$, $H_{ijk}$, $h_{ijkm}$ respectively set to 30 cm, 22 m and 5 m, the function $f(\mathbf{x}_{ijkm},\boldsymbol{\beta},\mathbf{0})$ yields a value of 551.3008 cm$^2$. The calculation of the matrix with the above derivatives C3a, C3b, and C3c yields the following matrix:

$$\mathbf{Z}'_{ijkm} = \begin{bmatrix}-1288.421 & 462.174\\ 462.174 & 1000.394\end{bmatrix}.$$

From C4, the approximate mathematical expectation of the squared diameter is

$$\begin{aligned}\hat{d}^2_{ijkm} &= 551.3008 + \frac{1}{2}tr\left(\mathbf{Z}'_{ijkm}\left(\begin{bmatrix}2.686\times10^{-2} & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 5.565\times10^{-3}\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 6.659\times10^{-3}\end{bmatrix}\right)\right).\\ &= 540.1095\end{aligned}$$

Note that the approximate expectation is smaller than the prediction conditional on the expectation of the random effects.